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Statistical Bootstrap Model of Hadrons*

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The hadron is considered to be a compound with two or more constituents circulating freely in a box of radius $\approx 10^{-13}$ cm. The density of hadron levels, $\rho(m)$, is estimated from the number of states in the box (statistical condition) and is also required to be consistent with the spectrum of constituents, which are assumed to be the hadrons themselves (bootstrap condition). This type of model was first considered by Hagedorn, who obtained a solution of form $\rho(m) \sim cm^a e^{bm}$ with $a = -\frac{5}{2}$ which satisfied the bootstrap condition asymptotically to within a power of m. We obtain a solution with $a < -\frac{5}{2}$ which satisfies the bootstrap condition exactly in the high-mass limit. The constituents in the box are distributed with probability $P(n) = (\ln 2)^{n-1}/(n-1)!$; i.e., an average high-mass resonance decays (in the first generation of its decay chain) to two hadrons (69% probability) or three (24% probability). We also review briefly the thermodynamic applications of this model to high-energy scattering and astrophysics.

I. INTRODUCTION AND DISCUSSION OF MODEL

In 1936, Bethe[1] proposed a statistical model for the density of excited nuclear levels.[2] He considered a free fermion gas of Z protons and $(A-Z)$ neutrons in a box with the normal nuclear radius. That is, the potential was used only to provide the walls of the box; residual nucleon-nucleon interactions inside the box were neglected.

As the energy is raised above the Fermi level, the number of nuclear states increases rapidly. Already the first excited single-particle level can be filled in a number of ways by raising any of the nucleons near the top of the Fermi sea; each different way leaves a different hole behind and therefore a different state. The first two excited single-particle levels can be filled in an even larger number of ways by raising any two of the nucleons near the top of the Fermi sea, and so forth.

Studying this problem quantitatively, Bethe found that for excitation energies E such that most of the fermions are still degenerate, the density of states

$$\rho(E) \equiv dn/dE \tag{1.1}$$

in the box rises as

$$\rho(E) \propto e^{b\sqrt{AE}}, \tag{1.2}$$

where b is a numerical constant. Experimentally, excited nuclear levels show up as resonances. When they are counted, a rapid rise qualitatively consistent with Bethe's formula is found.[2]

Of course, since Bethe's model is statistical and the potential has been grossly oversimplfied, Eq. (1.2) often fails to fit specific nuclei in detail, especially for low excitations. The model i has subsequently been modified[2] by adding effects of the potential which distinguish between even and odd nuclei, by putting in some shell-model corrections, etc. These modifications improve the fit to specific nuclei.

In the present paper, we consider an analogous model for hadrons. Just as the nucleus is considered to be a compound with A constituents drawn from two varieties (n and p), we consider the hadron to be a compound with $n \geq 2$ constituents drawn from various varieties (e.g., the three varieties of quark in the quark model, or many varieties of hadron in the bootstrap model). The potential is used explicitly only to define the walls of the box, with a radius of order 10^{-13} cm, since we know hadron structure is confined within a distance of this order. Inside the box, constituents will circulate without interacting. The density of levels in the box will be identified with the number of hadron states (as listed, for example, in the Particle Properties Tables) per unit interval of rest mass. Of course, this is the crudest dynamics possible, but it has the virtues of being soluble and of treating all states on the same footing (very important for the bootstrap case). The dynamics can be improved later if some detailed effects of the potential are understood, just as Bethe's free-fermion-gas model was later improved.

This approach to hadrons, and the systematic method of analysis we shall follow, were first introduced by Hagedorn in 1965.[3] We shall rephrase his analysis in rather different language and introduce some technical changes, to be noted below.

The mathematical counting of states in the box proceeds as follows:

For *one particle*, the density of states inside the box goes like

$$Vd^3p_i/h^3.$$

For *n independent particles* with total energy m, it is

$$\frac{d(\text{number of states})}{dm}$$

$$\equiv \rho_n(m)$$

$$=\left(\frac{V}{h^3}\right)^{n-1}\prod_{i=1}^{n}\int d^3p_i\,\delta\left(\sum_{i=1}^{n}E_i-m\right)\delta^3\left(\sum_{i=1}^{n}\vec{\mathrm{p}}_i\right). \quad (1.3)$$

Here we have counted only the density of levels with center of mass at rest, because this is the density to be identified with the number of hadron states per unit interval of rest mass.

As preliminary examples, we consider several simple models for the constituents:

(*i*) *Quark-antiquark model of mesons*. In its naive form, this model has $n=2$. The integral (1.3) is trivial and one finds[4]

$$\rho(m)\sim m^2. \quad (1.4)$$

(*ii*) *Three-quark model of baryons*. Here, $n=3$. The extra $\int d^3p_i$ increases the density of states to

$$\rho(m)\sim m^5. \quad (1.5)$$

(*iii*) *Single-elementary-particle model of mesons*. Suppose there were a single elementary boson x (let us ignore, for the purposes of this example, such complications as spin, charge, and strangeness). Suppose mesons were made of xx pairs, xxx triplets, $xxxx$ quartets, etc. – i.e., $n=2,3,\ldots,\infty$. In this case the density of states would be

$$\rho(m)=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int d^3p_i\,\delta\left(\sum_{i=1}^{n}E_i-m\right)\delta^3\left(\sum_{i=1}^{n}\vec{\mathrm{p}}_i\right), \quad (1.6)$$

where the factor $1/n!$ appears because only totally symmetric states of n bosons can be counted. The integrals in Eq. (1.6) can be evaluated approximately, yielding

$$\rho(m)\sim\exp(bm^{3/4}), \quad (1.7)$$

a much more rapid growth than in previous examples because states of all n are now included. Although the evaluation of (1.6) is well known[5] and elementary [for example, the number of photon states in an enclosed space is described by an expression similar to (1.6)], we present it in the Appendix for completeness.

The model we wish to focus on in this paper is the *bootstrap model of hadrons*, in which the hadrons are assumed to be compounds of hadrons. The model can be represented schematically by

$$\underset{\rho_{\text{out}}(m)}{\begin{pmatrix}\pi\\ K\\ \eta\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}}=\overset{n=2}{\begin{pmatrix}\pi\pi\\ K\pi\\ KK\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}}+\overset{n=3}{\begin{pmatrix}\pi\pi\pi\\ K\pi\pi\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}}+\overset{n=4}{\begin{pmatrix}\pi\pi\pi\pi\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}}+\cdots.$$

The equation for the density of states is

$$\rho_{\text{out}}(m)=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i\,\rho_{\text{in}}(m_i)$$
$$\times\int d^3p_i\,\delta\left(\sum_{i=1}^{n}E_i-m\right)\delta^3\left(\sum_{i=1}^{n}\vec{\mathrm{p}}_i\right), \quad (1.8)$$

which can be explained as follows:

(i) The integral over mass appears on the right-hand side because each particle in the box can take on not only different states of motion with phase space d^3p_i, but also different states of mass with density labeled by $\rho_{\text{in}}(m_i)$. Included in the single-particle density $\rho_{\text{in}}(m_i)$ are all different states of spin, charge, strangeness, baryon number, etc. – for example, π is counted as $(2I+1)=3$ states, ρ as $(2I+1)(2S+1)=9$ states, and so forth.

(ii) The factor $1/n!$, which was required for states consisting of n identical particles, is also needed for states consisting of nonidentical particles to avoid double counting.[6]

(iii) There is one error in (1.8): Configurations with two or more identical fermions in the same state are counted. The resulting overestimate of phase space should be slight because states containing fermion pairs are expected to be statistically unimportant in the hadron spectrum. The author has checked this by repeating the calculation using the partition function, which does take the Pauli principle into account; the leading asymptotic results discussed in this paper were essentially unchanged. Of course, for problems involving degeneracy (nuclei, neutron stars, etc.), it is crucial to take the Pauli principle into account and Eq. (1.8) would not be appropriate.

(iv) On the left-hand side of (1.8), we introduce the notation $\rho_{\text{out}}(m)$ for the total density of states in the box, again counting all quantum numbers. In a complete bootstrap theory, $\rho_{\text{out}}(m)$ would be the same as $\rho_{\text{in}}(m)$, but in any approximate model such as ours it is not possible to make them consistent over the entire spectrum, and we must keep the separate labels. The best we can do is

$$\rho_{\text{out}}(m)\xrightarrow[m\to\infty]{}\rho_{\text{in}}(m). \quad (1.9)$$

At low m, our statistical approach cannot hope to give the exact self-consistent $\rho(m)$.

The density of states (1.8) and the bootstrap condition (1.9) define our version of the bootstrap model. From the previous example (iii) of a single variety of input particle, we know that if $\rho_{\text{in}}(m) = \delta(m - m_0)$, then $\rho_{\text{out}}(m) \sim \exp(bm^{3/4})$. So $\rho_{\text{out}}(m)$ must grow at least this fast. We analyze Eq. (1.8) mathematically in Sec. II and show that to satisfy (1.9),

(i) $\rho_{\text{in}}(m)$ must increase faster than $\exp(bm^{1-\epsilon})$, where ϵ is any positive number greater than zero (otherwise ρ_{out} grows exponentially faster than ρ_{in}).

(ii) $\rho_{\text{in}}(m)$ must increase slower than $\exp(bm^{1+\epsilon})$, where ϵ is any positive number greater than zero (otherwise ρ_{out} grows exponentially slower than ρ_{in}).

(iii) There are various solutions increasing essentially as fast as e^{bm}; in particular,

$$\rho_{\text{in}}(m) = cm^a e^{bm}, \quad a < -\tfrac{5}{2} \tag{1.10}$$

satisfies (1.9) fully (with one numerical constraint on c), while

$$\rho_{\text{in}}(m) = c'm^{-5/2} e^{bm} \tag{1.11}$$

satisfies (1.9) as far as the exponential behavior is concerned, but not in the power ($a_{\text{out}} > a_{\text{in}}$). We shall devote most of our attention to the fully consistent solution (1.10).

Historically, a number of quite different models have yielded hadron densities growing as $cm^a e^{bm}$. The first was Hagedorn's statistical model[3] of 1965. Hagedorn invented the systematic argument we have followed above, and introduced the principal applications[7-10] which have been made up to the present time.

The specific solutions $cm^a e^{bm}$ obtained by Hagedorn were asymptotically self-consistent only in the exponent; for the power behavior, he found[3] one solution with

$$a_{\text{in}} < -\tfrac{5}{2}, \quad a_{\text{out}} = a_{\text{in}} + \tfrac{3}{2}, \tag{1.12}$$

and another with

$$a_{\text{in}} = -\tfrac{5}{2}, \quad a_{\text{out}} > -1\,. \tag{1.13}$$

The second solution was favored. The reasons for the slight difference between Hagedorn's solutions and ours are as follows[11]:

(i) Hagedorn included one-particle states in the box ($n = 1, 2, \ldots, \infty$). This is the normal procedure in statistical mechanics, but we prefer to consider only $n \geq 2$ because we are interested in the "compound states." The practical consequence of including single-particle states on the right-hand side of Eq. (1.8) is to make it impossible to satisfy the bootstrap condition exactly; ρ_{out} is always greater than ρ_{in}.

(ii) Hagedorn effectively included the states of motion of the center of mass. We do not wish to do so because it is the density of states at rest in the center-of-mass system that we associate with the number of hadron states per unit rest-mass interval. The practical consequence of including the phase space Vd^3p/h^3 of the center of mass is to make ρ_{out} grow faster, relative to ρ_{in}, by a factor $m^{3/2}$ [it is shown in Sec. II that the important contributions are nonrelativistic; here $p = (2mE_{\text{kin}})^{1/2}$, so $\int d^3p \sim m^{3/2}$]. This shift of $\frac{3}{2}$ in the power is precisely the difference between Hagedorn's solutions [(1.12) and (1.13)] and ours [(1.10) and (1.11)].

There is also a difference in language; we have worked directly with the density of states $\rho(m)$ whereas Hagedorn worked with the partition function[12]

$$Z(T) = \int^{\infty} dm\ \rho(m) e^{-m/T}. \tag{1.14}$$

Actually (1.14) is just the Laplace transform of $\rho(m)$ and the results obtained working with ρ can be obtained working with Z [provided points (i) and (ii) of the previous paragraph are treated appropriately]. The reason we choose to work directly with the density of states, at the cost of some awkwardness with the Pauli exclusion principle, is mainly pedagogical: Phase space is more familiar to particle physicists, and the independence of the derivation from any assumptions concerning thermal equilibrium is most convincingly demonstrated by not introducing temperature at all.

The most famous applications of Hagedorn's hadron spectrum are related to the exponential growth, and will still apply. There are interesting consequences of taking the power $a < -\frac{5}{2}$ instead of $a = -\frac{5}{2}$, however, which we shall note as we go along.

Subsequent to Hagedorn's work, completely different approaches based on duality[13,14] and the Veneziano representation[15-18] also yielded the spectrum $\rho \sim cm^a e^{bm}$. Factorization of the N-point Veneziano representation even yielded a similar set of possibilities for a,[19]

$$a = -\tfrac{5}{2},\ -3,\ -\tfrac{7}{2}, \ldots\,. \tag{1.15}$$

In Sec. V we discuss a possible interpretation of this remarkable correspondence: The number of resonances in the statistical model grows at the same rate as the number of open channels, and this allows one to represent a typical amplitude as a sum over direct-channel resonances, which is a necessary condition for duality.[20]

In Sec. III we consider further the particle spectrum, including the spectrum of hadrons with *specific* baryon number, strangeness, charge, etc.

In Sec. IV we study the predictions of the model concerning statistically dominant couplings. The average number of particles coupled to, which was $\bar{n} \sim \ln m$ in Hagedorn's case ($a = -\frac{5}{2}$), is $\bar{n} = 2.4$ in our case ($a < -\frac{5}{2}$), independent of mass (this is the "first-generation" coupling of a resonance, not the result of the entire decay chain). In detail, the probability for a resonance to couple to different numbers of secondaries is

$$P_n = \frac{(\ln 2)^{n-1}}{(n-1)!}\,, \tag{1.16}$$

i.e., 69% to two-body channels and 24% to three-body channels, independent of mass. Unfortunately, our knowledge of decay systematics of heavy resonances is insufficient to check these predictions, but our result on $\bar{n}$ bears a striking resemblance to most dynamical models, which usually couple resonances directly to only two or three particles.

Finally, in Sec. VI we review, for completeness, highlights of Hagedorn's fascinating work[7-10] on the thermodynamics associated with this model. We point out that in high-energy collisions, the model does not permit establishment of thermal equilibrium with a uniform temperature over the entire interaction volume (as in the old Fermi statistical model). Establishment of local thermal equilibrium, with temperature varying over the interaction volume (as in Hagedorn's model of collisions), is permitted provided $a > -\frac{7}{2}$. With regard to recent attempts to apply the spectrum $cm^a e^{bm}$ to neutron stars and the "big bang,"[10,19] we emphasize that they involve an additional element of speculation: Just when the high-mass hadron spectrum becomes relevant, the hadrons are squeezed together and overlapping in space, in which case our derivation of the spectrum in terms of an isolated box of radius $\approx 10^{-13}$ cm may fail.

In reading this paper, note that the messy and approximate mathematical study of phase space is confined to Sec. II and the Appendix. The reader uninterested in such details may skip these sections or, better, read only the portions $p = 1$ and $p = 1$, $a < -\frac{5}{2}$ of Sec. II which deal with the self-consistent case.

II. MATHEMATICAL STUDY OF BOOTSTRAP CONDITIONS

In the present section, we find solutions of the bootstrap condition

$$\rho_{\text{out}}(m) \xrightarrow[m\to\infty]{} \rho_{\text{in}}(m)\,, \tag{1.9}$$

where ρ_{out} is given by the phase-space integral

$$\rho_{\text{out}}(m) = \sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i\,\rho_{\text{in}}(m_i) \times \int d^3p_i\,\delta\Big(m - \sum_{i=1}^{n}E_i\Big)\delta^3\Big(\sum_{i=1}^{n}\vec{p}_i\Big). \tag{1.8}$$

The ρ's, of course, satisfy $\rho(m) = 0$ at $m < m_0$ and $\rho(m) \geq 0$ at $m \geq m_0$, where m_0 is the mass of the lightest hadron.

We shall show that at large m

(i) ρ increases faster than $\exp(bm^{1-\epsilon})$, where ϵ is any positive number greater than zero.

(ii) ρ increases slower than $\exp(bm^{1+\epsilon})$, where ϵ is any positive number greater than zero.

(iii) Thus, any solution must grow essentially as e^{bm}. We do not find a unique solution or a complete list of solutions, but we do show specifically that $\rho(m) = cm^a e^{bm}$, with $a < -\frac{5}{2}$, satisfies conditions (1.8) and (1.9).

It has already been shown that if only one variety of particle is put into the box $[\rho(m) = \delta(m - m_0)]$, the output spectrum grows as $\exp(bm^{3/4})$. Thus we are assured that ρ grows at least as fast as $\exp(bm^{3/4})$, and we consider ρ_{in} of the form

$$\rho_{\text{in}}(m) = f(m)\exp(bm^p)\,, \tag{2.1}$$

where $f(m)$ and $f^{-1}(m)$ are polynomial-bounded at large m.

<u>$p < 1$.</u> Inserting (2.1) into (1.8), we find for $p < 1$ that the integrand of the nth term is maximal when all the m_i are equal and of order m/n. The exponential factors dominate, and at $m_i = m/n$ they give an integrand of order

$$\prod_{i=1}^{n}\exp[b(m/n)^p] = \exp(n^{1-p}bm^p)\,. \tag{2.2}$$

The density $\rho_{\text{out}}(m)$ grows at least as fast as the exponential in (2.2), which for $p < 1$ grows faster than $\rho_{\text{in}} \sim \exp(bm^p)$. Thus the bootstrap condition cannot be satisfied with $p < 1$.

Note that this reasoning (with the exponential mass factors dominant) applies to all n for which a sufficiently large energy is available for each particle in the box. It certainly applies at sufficiently large m when $n \leq Cm^{1-\epsilon}$, with $\epsilon > 0$, for then each particle has an energy $\sim m^\epsilon$ available. This means that the exponential in (2.2) can be at least as large as

$$\exp(n^{1-p}bm^p) \sim \exp[C'm^{1-\epsilon(1-p)}]\,, \tag{2.3}$$

which shows directly that for $\rho_{\text{in}} \sim \exp(bm^p)$, ρ_{out} grows faster than $\exp(bm^{1-\epsilon})$, where ϵ is any positive number.

<u>$p > 1$.</u> At $m_i = m/n$, Eq. (2.2) shows that the integrand grows exponentially *slower* than ρ_{in} when $p > 1$. In fact, the integrand for the nth term reaches its maximum, in this case, at the bound-

ary where one m_i is as large as possible and all others are as small as possible. At this point one obtains

$$\prod_{i=1}^{n}\exp(bm_i{}^p)=\exp\{b[m-(n-1)m_0]^p\}\exp[(n-1)bm_0{}^p]$$
$$\sim\exp(bm^p)\exp[-bp(n-1)m_0m^{p-1}]\,, \tag{2.4}$$

which still grows exponentially slower than $\rho_{\text{in}}(m)$ because there is no massless hadron. In the n-particle term, the exponential (2.4) is multiplied only by a polynomial-bounded function of m. Thus the nth term increases more slowly than the $(n-1)$th term, by a factor of order $\exp(-bpm_0m^{p-1})$. The dominant term is $n=2$, but even this contribution to ρ_{out} grows more slowly than ρ_{in} by

$$\exp(-bpm_0m^{p-1})\,.$$

Thus the bootstrap condition cannot be satisfied with $p>1$.

In Hagedorn's formulation[3] the situation was somewhat different. An $n=1$ contribution to ρ_{out} was included. This grew as fast as ρ_{in}, so solutions with $p>1$ had to be ruled out by an appeal to thermodynamics.

$p=1.$ We now specialize to the form

$$\rho_{\text{in}}(m)=cm^ae^{bm}\,, \tag{2.5}$$

though a series of terms of this form, or of such terms multiplied by $(\ln m)^r$, could equally well be considered. Inserting (2.5) into (1.8), one obtains

$$\rho_{\text{out}}(m)=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i cm_i{}^ae^{bm_i}$$
$$\times\int d^3p_i\,\delta\left(\sum_{i=1}^{n}E_i-m\right)\delta^3\left(\sum_{i=1}^{n}\vec{p}_i\right). \tag{2.6}$$

It is useful to recast the exponential factors in terms of the kinetic energy Q_i of the ith particle ($E_i=m_i+Q_i$). With the help of energy conservation we find

$$\prod_{i=1}^{n}e^{bm_i}=\exp\left(b\sum_i m_i\right)=\exp\left[b\left(m-\sum_i Q_i\right)\right]$$
$$=e^{bm}\prod_{i=1}^{n}e^{-bQ_i}\,. \tag{2.7}$$

Equation (2.6) now takes the form

$$\rho_{\text{out}}(m)=ce^{bm}\sum_{n=2}^{\infty}\left(\frac{cV}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_im_i{}^a$$
$$\times\int d^3p_ie^{-bQ_i}\delta\left(\sum_i E_i-m\right)\delta^3\left(\sum_i\vec{p}_i\right), \tag{2.8}$$

which shows that e^{bm} at least reproduces itself. Equation (2.8) also shows that large kinetic energies are exponentially damped, most of the energy going into mass.

The behavior of Eq. (2.8) is fundamentally different for $a>-\frac{5}{2}$ and $a<-\frac{5}{2}$. To show this, we consider the relation between momentum and kinetic energy,

$$p_i{}^2=E_i{}^2-m_i{}^2=(m_i+Q_i)^2-m_i{}^2=Q_i{}^2+2Q_im_i\,. \tag{2.9}$$

Because the momentum integration is effectively cut off at $Q_i\approx b^{-1}$, large m_i are important for the terms $n\leqslant Cm^{1-\epsilon}$, $\epsilon>0$. For large m_i, we can use the nonrelativistic expression

$$p_i\approx(2Q_i\,m_i)^{1/2}. \tag{2.10}$$

Thus the integral for particle i has the approximate form

$$I_i=\int dm_i\,m_i{}^a\int d^3p_ie^{-bQ_i}\approx\frac{1}{b^{3/2}}\int dm_im_i{}^{a+3/2}\,. \tag{2.11}$$

The integral is cut off in practice by energy conservation; denoting the cutoff by Λ ($\Lambda\lesssim m$), we distinguish the cases

$$\begin{aligned} a&>-\tfrac{5}{2}, & I_i&\sim\Lambda^{a+5/2}\\ &=-\tfrac{5}{2}, & I_i&\sim\ln\Lambda\\ &<-\tfrac{5}{2}, & I_i&\ \text{convergent}\,.\end{aligned} \tag{2.12}$$

Let us take up these three cases in succession.

$p=1,\ a>-\frac{5}{2}.$ According to Eqs. (2.11) and (2.12), the integral over the masses and momenta available to any one particle is of order $b^{-3/2}\Lambda^{a+5/2}$. The contribution to the n-particle term is maximal (if $n\leqslant Cm^{1-\epsilon}$) when Λ for each particle is of order m/n. Thus the n-particle term is of order

$$\rho_{\text{out}}^{(n)}(m)\sim\frac{ce^{bm}}{n!}\left(\frac{cV}{h^3}\right)^{n-1}\left(\frac{m}{n}\right)^{(n-1)(a+5/2)}\left(\frac{1}{b}\right)^{3(n-1)/2}$$
$$=\frac{ce^{bm}}{n!}\left(\frac{cVm^{a+5/2}}{h^3b^{3/2}n^{a+5/2}}\right)^{n-1}$$
$$\equiv ce^{bm}e^{f(m,n)}\,, \tag{2.13}$$

where, with use of Stirling's formula,

$$f(m,n)\simeq-n\ln n+n+(n-1)\ln\left(\frac{cVm^{a+5/2}}{h^3b^{3/2}n^{a+5/2}}\right). \tag{2.14}$$

The sum over n can be approximated by an integral

$$\rho_{\text{out}}(m)=ce^{bm}\int dn\,e^{f(m,n)}\,, \tag{2.15}$$

in which the integrand is maximal at $\partial f/\partial n=0$, i.e., at

$$n_{\max}\approx\left[\frac{cV}{h^3b^{3/2}}\left(\frac{m}{e}\right)^{a+5/2}\right]^{1/(a+7/2)}. \tag{2.16}$$

This maximum is within the region $n\leqslant Cm^{1-\epsilon}$ where our estimates apply; the contribution from $n>Cm^{1-\epsilon}$ will not affect the following conclusions.

The value of f at its maximum is

$$f_{\max}=c'm^{(a+5/2)/(a+7/2)}; \tag{2.17}$$

thus

$$\rho_{\rm out}(m)\approx ce^{bm}\exp(c'm^{(a+5/2)/(a+7/2)}), \tag{2.18}$$

which increases exponentially faster than $\rho_{\rm in}$. So the bootstrap condition cannot be satisfied for $p=1,\ a>-\frac{5}{2}$.

$p=1,\ a=-\frac{5}{2}$. In this case, the integral over the masses and momenta available to any one particle is of order $b^{-3/2}\ln(\Lambda/m_0)$ [Eqs. (2.11) and (2.12)]. The contribution to the n-particle term is again maximal when $\Lambda\approx m/n$. The nth term is of order

$$\begin{aligned}\rho_{\rm out}^{(n)}(m)&\simeq c\left(\frac{m}{n}\right)^{-5/2}\frac{e^{bm}}{n!}\left(\frac{cV}{h^3}\right)^{n-1}\\&\quad\times\left(\ln\frac{m}{nm_0}\right)^{n-1}\left(\frac{1}{b}\right)^{3(n-1)/2}\\&=c\left(\frac{m}{n}\right)^{-5/2}\frac{e^{bm}}{n!}\left(\frac{cV\ln(m/nm_0)}{h^3b^{3/2}}\right)^{n-1},\end{aligned} \tag{2.19}$$

where the factor $(m/n)^{-5/2}$ is obtained by working out the consequences of energy-momentum conservation for the nth particle with more care than in the previous case (the numerical coefficient has still been approximated very crudely, however, since we will not make use of it). Proceeding as before, one easily finds that the maximum in the sum over n occurs at

$$n\approx\frac{cV}{h^3b^{3/2}}\ln\left(\frac{m}{nm_0}\right), \tag{2.20}$$

which is again well within the range $n\leq Cm^{1-\epsilon}$ where our approximations make sense. Dropping some logarithmic factors which are of purely secondary interest compared to the power behavior, we find the sum over n has the approximate form (with $c'=cV/h^3b^{3/2}$)

$$\begin{aligned}\rho_{\rm out}(m)&\approx cm^{-5/2}e^{bm}\sum_n\frac{(c'\ln m)^n}{n!}\\&=cm^{-5/2}e^{bm}e^{c'\ln m}\\&=cm^{-5/2+c'}e^{bm},\end{aligned} \tag{2.21}$$

confirming our earlier statement that $p=1,\ a=-\frac{5}{2}$ satisfies the bootstrap condition as far as the exponential is concerned, but not in the power behavior. The power $a_{\rm out}$ exceeds $-\frac{5}{2}$ by a term of order $c'=cV/h^3b^{3/2}$, i.e., by a numerical factor which depends on the parameters of the model.

If we had followed Hagedorn[3] and had included the phase space associated with center-of-mass motions, an extra factor

$$(V/h^3)(m_i/b)^{3/2}\simeq(V/h^3)(m/nb)^{3/2}$$

would have occurred. Equation (2.19) would have read

$$\begin{aligned}\rho_{\rm out}^{(n)}(m)&\simeq c\left(\frac{n}{m}\right)\frac{e^{bm}}{n!}\left(\frac{V}{h^3b^{3/2}}\right)\left(\frac{cV\ln(m/nm_0)}{h^3b^{3/2}}\right)^{n-1}\\&\simeq\frac{ce^{bm}}{m}\left(\frac{V}{h^3b^{3/2}}\right)\frac{(c'\ln m)^{n-1}}{(n-1)!},\end{aligned} \tag{2.22}$$

which sums to

$$\rho_{\rm out}(m)=\sum\rho_{\rm out}^{(n)}\cong cm^{-1+c'}e^{bm}\left(\frac{V}{h^3b^{3/2}}\right). \tag{2.23}$$

Here the discrepancy in the power a has grown by an additional $\frac{3}{2}$ as mentioned in Sec. I.

$p=1,\ a<-\frac{5}{2}$. In this case the contribution to the nth term comes mainly from the region where all p_i are small, one m_i is large, and the other m_i are small. It turns out that $n=2$ is the dominant term. Therefore, we begin with a careful evaluation of the $n=2$ term:

$$\rho_{\rm out}^{n=2}(m)=\frac{V}{2h^3}\prod_{i=1}^{2}\int dm_i\,cm_i{}^ae^{bm_i}\int d^3p_i\,\delta(E_i+E_2-m)\delta^3(\vec{p}_1+\vec{p}_2). \tag{2.24}$$

Four of the six momentum integrations can be done with the aid of the δ functions, and the remaining two angular integrations simply give 4π. So we obtain the exact relation

$$\begin{aligned}\rho_{\rm out}^{n=2}(m)&=\frac{\pi V}{4h^3m^4}\int_{m_0}^{m-m_0}dm_1\rho_{\rm in}(m_1)\int_{m_0}^{m-m_1}dm_2\rho_{\rm in}(m_2)\\&\quad\times(m^2+m_1{}^2-m_2{}^2)(m^2+m_2{}^2-m_1{}^2)\{[m^2-(m_1-m_2)^2][m^2-(m_1+m_2)^2]\}^{1/2}.\end{aligned} \tag{2.25}$$

Specializing to $\rho=cm^ae^{bm}$, changing to the variables $m_\pm=m_1\pm m_2$, and using the symmetry between positive and negative m_-, we rewrite (2.25) as

$$\rho_{\rm out}^{n=2}=\frac{\pi Vc^2}{h^3m^44^{a+1}}\int_{2m_0}^{m}dm_+e^{bm_+}(m^2-m_+{}^2)^{1/2}\int_0^{m_+-2m_0}dm_-(m_+{}^2-m_-{}^2)^a(m^2-m_-{}^2)^{1/2}(m^4-m_+{}^2m_-{}^2), \tag{2.26}$$

which is still exact. One easily sees that the integrand peaks exponentially at $m_1+m_2\approx m$, and that m_1 large and m_2 small, or vice versa (i.e., $|m_-|\approx m$) is favored by a power. This peak region is indicated in

Fig. 1. In view of the peaking, we can approximate $(m^2 - m_+{}^2)^{1/2}$ by $(2m)^{1/2}(m - m_+)^{1/2}$, $(m_+{}^2 - m_-{}^2)$ by $2m(m_+ - m_-)$, $(m^2 - m_-{}^2)^{1/2}$ by $(2m)^{1/2}(m - m_-)^{1/2}$, and $(m^4 - m_+{}^2 m_-{}^2)$ by $2m^3(m - m_-)$, obtaining

$$\rho_{\text{out}}^{n=2}(m) \approx \frac{\pi V c^2}{h^3 2^a} m^a \int_{2m_0}^{m} dm_+ e^{bm_+}(m - m_+)^{1/2} \int_0^{m_+ - 2m_0} dm_-(m_+ - m_-)^a (m - m_-)^{3/2} . \tag{2.27}$$

In the limit $b^{-1} \ll m_0$, m_+ is so near m that the second integral becomes

$$\int_0^{m_+ - 2m_0} dm_-(m - m_-)^{a-3/2} \approx \frac{(2m_0)^{a+5/2}}{-(a + \frac{5}{2})} , \tag{2.28}$$

and $\rho_{\text{out}}^{n=2}$ can be evaluated as

$$b^{-1} \ll m_0 : \ \rho_{\text{out}}^{n=2}(m) = \frac{(2\pi)^{3/2} c V m_0{}^{a+5/2}}{-(a + \frac{5}{2}) h^3 b^{3/2}} (cm^a e^{bm}) . \tag{2.29}$$

One can also evaluate the limit $b^{-1} \gg m_0$, obtaining $m^a e^{bm}$ with a somewhat different numerical coefficient. The estimate (2.29) should not be too bad for the experimental parameters[21] $b^{-1} \approx m_0$ (i.e., $b^{-1} = 160$ MeV, $m_0 = 140$ MeV). But the important point here is that $\rho_{\text{out}}^{n=2}$ *equals* ρ_{in} *times a numerical coefficient;* the power m^a as well as the exponent e^{bm} has reproduced itself. The exact value of the numerical coefficient is of somewhat secondary interest and is, in any case, sensitive to the low-mass spectrum which is not well represented by the asymptotic form $cm^a e^{bm}$.

The contributions from higher n are also maximal when one mass is large and all others are small. One gets approximately

$$\rho_{\text{out}}^{(n)}(m) = \frac{(cm^a e^{bm})}{(n-1)!} \left(\frac{(2\pi)^{3/2} c V m_0{}^{a+5/2}}{-(a + \frac{5}{2}) h^3 b^{3/2}} \right)^{n-1} . \tag{2.30}$$

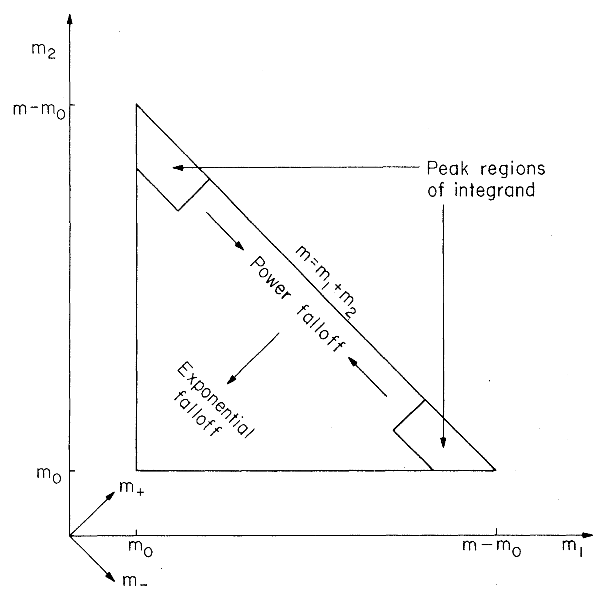

FIG. 1. The density of two-particle states, $\rho_{\text{out}}^{n=2}(m)$, as a function of m_1 and m_2.

Here $m^a e^{bm}$ comes from the one high-mass factor, as in the $n=2$ case. The factorial $1/(n-1)!$ comes from $1/n!$ times an n representing the fact that the integrand peaks when any one of the n particles has large m_i. There is a factor cV/h^3 for each new particle, a factor $m_i{}^{3/2}/b^{3/2}$ for each new $\int d^3 p_i$, with the $m_i{}^{3/2}$ getting absorbed into a factor $m_0{}^{a+5/2}/(-a - \frac{5}{2})$ for each new $\int dm_i m_i{}^{a+3/2}$. Finally, the $(2\pi)^{3/2}$ is our best estimate of the numerical coefficient. Again, the important point is that $\rho_{\text{out}}^{(n)}$ equals ρ_{in} times a numerical coefficient. Adding all terms, we obtain

$$\rho_{\text{out}}(m) = \rho_{\text{in}}(m) \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)!} , \tag{2.31}$$

$$x = \frac{(2\pi)^{3/2} c V m_0{}^{a+5/2}}{-(a + \frac{5}{2}) h^3 b^{3/2}} . \tag{2.32}$$

The bootstrap condition $\rho_{\text{out}} \to \rho_{\text{in}}$ requires

$$\sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 , \tag{2.33}$$

i.e.,

$$x = \ln 2 \simeq 0.69 . \tag{2.34}$$

This provides one constraint on the parameters of the model (a, b, c, V), leaving three free parameters. Concerning the detailed form of Eq. (2.32), however, we remind the reader once more that x is sensitive to the low-mass spectrum, so that (2.32) is not expected to be very accurate.

III. THE PARTICLE SPECTRUM

The relation of the particle spectrum $\rho \sim cm^{-5/2}e^{bm}$ to the data (as tabulated in the Particle Properties Tables) has been studied by Hagedorn.[21] The experimental $\rho(m)$ rises rapidly (Fig. 2) and is consistent with theory up to energies where our detailed knowledge of the particle spectrum becomes seriously incomplete. This is encouraging, but does not prove the model is correct, since other rapidly rising functions of m can also be devised which fit the data with a couple of adjustable parameters. By the same token, it is not possible to distinguish between $a = -\frac{5}{2}$ and a somewhat less than $-\frac{5}{2}$.

Up to this point we have discussed the over-all ρ counting all hadrons. It is possible to make similar theoretical arguments concerning the spectrum $\rho_{BSQ}(m)$ for *specific* quantum numbers, such as

$B=1$, $S=0$, or $B=0$, $S=1$, etc. The formulas for $\rho_{BSQ}(m)$ in our model are

$$\rho_{BSQ\,\mathrm{out}}(m)=\sum_{n=2}^{\infty}\left(\frac{V}{h^3}\right)^{n-1}\frac{1}{n!}\prod_{i=1}^{n}\int dm_i\sum_{B_iS_iQ_i}\rho_{B_iS_iQ_i\,\mathrm{in}}(m_i)$$
$$\times\int d^3p_i\,\delta\left(\sum_{i=1}E_i-m\right)\delta^3\left(\sum_{i=1}\vec{p}_i\right)\delta\left(\sum_i B_i-B\right)$$
$$\times\,\delta\left(\sum_i S_i-S\right)\delta\left(\sum_i Q_i-Q\right) \qquad (3.1)$$

and

$$\rho_{BSQ\,\mathrm{out}}(m)\underset{m\to\infty}{\rightarrow}\rho_{BSQ\mathrm{in}}(m)\,, \qquad (3.2)$$

plus the conditions $\rho_{BSQ}(m<m_{0,BSQ})=0$ and $\rho_{BSQ}(m>m_{0,BSQ})\geqslant 0$, where $m_{0,BSQ}$ is the mass of lightest hadron with quantum numbers B, S, and Q. In practice, at any m, only a finite number of B_i, S_i, and Q_i contribute, of order

$$|B_i|\lesssim m/m_N\,,$$
$$|S_i|,\ |Q_i|\lesssim m/m_\pi\,. \qquad (3.3)$$

Thus, for

$$\rho(m)=\sum_{B,S,Q}\rho_{BSQ}(m) \qquad (3.4)$$

to hold with $\rho(m)\sim cm^ae^{bm}$, we must have at least

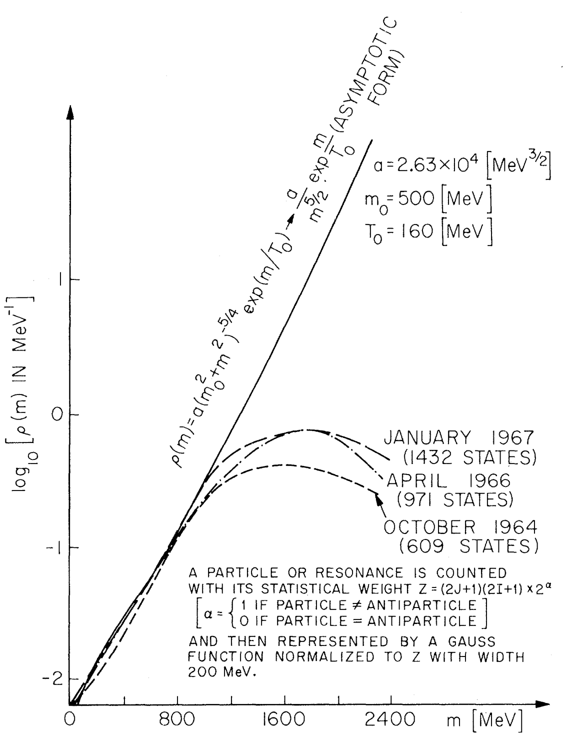

FIG. 2. The smoothed experimental mass spectrum as it developed from 1964 to 1967, compared with the function $\rho(m)=a(m_0^2+m^2)^{-5/4}\exp(m/T_0)$, which has the asymptotic form required by Hagedorn's theory. The figure is taken from Hagedorn, Ref. 21.

one $\rho_{B_iS_iQ_i}(m)$ as large as $(m_\pi/m)^3cm^ae^{bm}$ at each m. Consider the $n=2$ contribution from this particular $\rho_{B_iS_iQ_i}$ and its antiparticle density $\rho_{-B_i,-S_i,-Q_i}$ to $\rho_{000,\mathrm{out}}(m)$ in Eq. (3.1). It is of order $\rho_{000}\sim c'm^{a'}e^{bm}$, where a' is some power not exceeding a. Next, consider the $n=2$ contribution to each $\rho_{BSQ}(m)$ from $\rho_{000}(m_1\approx m-m_{0,BSQ})$ and $\rho_{BSQ}(m_2\approx m_{0,BSQ})$. By the arguments of Sec. II, it gives $\rho_{BSQ}(m)\sim c'_{BSQ}m^{a'}e^{bm}$. If, from summing over other contributions, some $\rho_{B_iS_iQ_i}$ grows with a higher power $a''>a'$, we can go back and consider again the $n=2$ contribution to $\rho_{000}(m)$ from $\rho_{B_iS_iQ_i}$(large m_1) and $\rho_{-B_i,-S_i,-Q_i}$ (small m_2), with the result that ρ_{000} also grows with the higher power. Proceeding in this way, we find that all ρ_{BSQ} grow asymptotically as

$$\rho_{BSQ}(m)\sim c_{BSQ}m^{a''}e^{bm}\,, \qquad (3.5)$$

where a'' is a common power.

Determination of c_{BSQ} and $(a''-a)$ from the coupled equations (3.1), (3.2), and (3.4) is an interesting problem which we shall leave for future consideration, since it seems to be sensitive to "transient" terms which die away at large m but are important near threshold. Experimentally, the density of hadron states with the quantum numbers of π, K, N, and Y, respectively, rises rapidly above their thresholds, in a manner similar to the total density.[8]

Our model also gives hadron states in "exotic channels" such as $B=1$, $S=1$ or $B=0$, $Q=2$ since, for example, K^+p and $\pi^+\pi^+$ states are counted. At present, no particles with these quantum numbers have been firmly established. Dynamical models[22] indicate that the potentials for these states are relatively weak or possibly repulsive, unlike the strongly attractive potentials for nonexotic states. Evidently, to fit the data in detail we must include these differences in the potential for various states, just as to fit the nuclear data in detail one includes effects of the nuclear potential.[2] Of course, once two-body potentials are introduced, the model becomes less unique. The result might be that exotic states are not confined inside the box at all. Or perhaps there is a repulsive potential which simply raises the "floor" of the box on the order of 1 BeV for each pair with exotic quantum numbers. In the latter case, exotic resonances would occur once the mass was higher than the "floor," and their level density would rise asymptotically in the usual fashion $\rho_{BSQ}\sim c_{BSQ}m^{a''}e^{bm}$.

The reader may ask why one should not include attractive potentials for "nonexotic" channels as well. The answer is that the main effects of attractive potentials have already been included implicitly, not only in the walls of the box but also in the resonance states. To see this, consider two particles which attract each other moving around in a box. As a result of the attraction, the wave function involving the relative coordinate of the two particles oscillates more rapidly than usually when the particles are close together. The more rapid oscillation means that more states fit into the box; specifically, when there is one extra oscillation (phase shift of 180°), one extra state can be fitted into the box. In this case, although an exact calculation would count states of motion of the original two particles with their mutual potential, it is approximately valid to omit the potential and count states of motion of the original two particles (treated as noninteracting) *plus* states of motion of the resonance.[23] This is what we have been doing by counting all resonances as independent particles.

IV. STATISTICALLY DOMINANT COUPLINGS

Let us consider the decay of a typical resonance of mass m. Assume that the transition rates to the various open channels are proportional to phase space, which may be sensible at high mass where a large number of channels are open. Since the interaction radius is of the same order as the hadron radius, we may use the phase space in our box to discuss decays. We can take over directly the results of Sec. II for the phase space as a function of the number, masses, and momenta of the particles in the box.

In practice, a heavy resonance decays into lighter resonances, which themselves decay, forming a chain which eventually leads to a final state containing light metastable particles such as pions. Since we are counting all channels including those with high-mass (unstable) particles, we shall only obtain statements about the first generation in the chain. In the present paper, we shall not consider the more complex question of what emerges at the end of the whole chain. Thus we will not be able to make direct comparisons with presently available data, though our results on "first-generation" couplings can be compared with theoretical vertices appearing in dynamical schemes such as the Veneziano model.

First, consider the number of particles produced in the first-generation decay. It was shown in Eq. (2.20) that for $a=-\frac{5}{2}$ (Hagedorn's model), phase space peaks at $\bar{n}\sim\ln m$. On the other hand, for $a<-\frac{5}{2}$, phase space is maximal at $n=2$ and is distributed according to Eqs. (2.31) and (2.34) with probabilities

$$P(n)=\frac{(\ln 2)^{n-1}}{(n-1)!}, \tag{4.1}$$

independent of mass. Numerically, $P(2)\approx 0.69$, $P(3)\approx 0.24$, and $P(4)\approx 0.06$, i.e., the 2- and 3-particle channels dominate very strongly. The average particle number is

$$\begin{aligned}\bar{n}&=\sum_{n=2}^{\infty} nP(n)\\ &=\frac{d}{dx}\left(x\sum_{n=2}^{\infty}\frac{x^{n-1}}{(n-1)!}\right)\bigg|_{x=\ln 2}\\ &=\frac{d}{dx}(xe^{x}-x)\bigg|_{x=\ln 2}\\ &=1+2\ln 2\approx 2.4\,.\end{aligned} \tag{4.2}$$

To the best of the author's knowledge, this is the first time that the theoretically fashionable procedure of coupling resonances mainly to two-body or three-body channels has been justified by a systematic argument.

It was also shown in Sec. II that for $a<-\frac{5}{2}$, phase space peaks when one particle has most of the mass and the other particles have as little mass as possible. The phase-space contribution for each low-mass particle is nearly independent of that for the others. These facts suggest a physical interpretation of Eq. (4.1): It is a modified Poisson distribution in the $(n-1)$ low-mass particles (with $n-1=0$, of course, omitted). The low-mass particles are emitted almost independently because each carries off only a small fraction of the total energy.

The other characteristics of heavy-resonance decay are easily read off from Sec. II. Large kinetic energies Q_i are damped by the factor $\exp(-bQ_i)$ [Eq. (2.8)] for both $a=-\frac{5}{2}$ and $a<-\frac{5}{2}$. To put it another way, most of the available energy goes into mass rather than kinetic energy,

$$m\geqslant\sum_{i=1}^{n} m_i \gtrsim m-\frac{n}{b}. \tag{4.3}$$

For $a=-\frac{5}{2}$, individual masses are distributed as dm_i/m_i, and the n particles tend to divide up the energy evenly. For $a<-\frac{5}{2}$, as already mentioned, one particle gets nearly all the mass; the other masses are small and distributed as $m_i^{a+3/2}dm_i$. These features are shown in Fig. 1 for the particular case $n=2$, $a<-\frac{5}{2}$.

One also finds that production of *specific* heavy particles, such as $p\bar{p}$ pairs, in resonance decay is damped exponentially because of the statistical competition.[8]

V. CONNECTION BETWEEN DUALITY AND STATISTICAL MODEL

Dual models,[13,14] some forms of the Veneziano model,[15-19] and the bootstrap statistical model[3] all lead to hadron spectra of type $\rho(m)\sim cm^{a}e^{bm}$. In the present section, we discuss a possible reason for this remarkable correspondence.[20]

One of the features of dual and Veneziano models is that a typical amplitude (specifically, a nonexotic amplitude which is not dual to the Pomeranchukon) can be represented as a sum over direct-channel resonances, even at arbitrarily high energies. From an over-all point of view, this implies that the supply of resonances in each mass interval, multiplied by the average resonance width $\overline{\Gamma}_{\text{tot}}(m)$, must be comparable to the number of channels open at each center-of-mass energy:

$$N_c(m) \propto \rho(m)\overline{\Gamma}_{\text{tot}}(m) . \tag{5.1}$$

If we make the usual assumption, consistent with existing data, that $\overline{\Gamma}_{\text{tot}}$ varies only slowly (at most as a power) with mass, then the requirement becomes essentially $N_c(m) \propto \rho(m)$.

Let us calculate the number of two-body channels open in the statistical model. At center-of-mass energy m, the number is

$$N_c^{n=2}(m) = \frac{1}{2!}\int_{m_0}^{m-m_0} dm_2 (cm_2{}^a e^{bm_2}) \times \int_{m_0}^{m-m_2} dm_1 (cm_1{}^a e^{bm_1}) , \tag{5.2}$$

where 1/2! is inserted to avoid double counting the ij and ji channels. Equation (5.1) is similar to the expression for two-body phase space. In terms of the variables $m_\pm = m_1 \pm m_2$, (5.2) becomes

$$N_c^{n=2}(m) = \frac{c^2}{2^{2a+1}}\int_{2m_0}^{m} dm_+ e^{bm_+} \times \int_0^{m_+-2m_0} dm_- (m_+{}^2 - m_-{}^2)^a . \tag{5.3}$$

For our case ($a \leq -\frac{5}{2}$), the integrand peaks strongly at $m_+ \approx m$, $m_- \approx m_+$ (i.e., m_1 large, and m_2 small, or vice versa) so we may approximate $(m_+{}^2 - m_-{}^2)^a$ by $(2m_+)^a(m_+ - m_-)^a$. The integral then gives

$$N_c^{n=2}(m) \approx \frac{cm^a e^{bm}}{b}\left(\frac{cm_0{}^{a+1}}{-(a+1)}\right). \tag{5.4}$$

Similarly,

$$N_c^n(m) \approx \frac{cm^a e^{bm}}{b(n-1)!}\left(\frac{cm_0{}^{a+1}}{-(a+1)}\right)^{n-1} \tag{5.5}$$

and

$$N_c^{\text{total}}(m) \approx \frac{\rho(m)}{b}\sum_{n=2}\frac{1}{(n-1)!}\left(\frac{cm_0{}^{a+1}}{-(a+1)}\right)^{n-1} . \tag{5.6}$$

Thus the number of open channels in the statistical model (with either $a = -\frac{5}{2}$ or $a < -\frac{5}{2}$) rises exactly in parallel with the number of resonances – just what is needed for duality. For comparison, the reader can easily verify that this condition is generally not satisfied by other expressions for the level density, such as $\rho(m) \sim m^p$, which are not solutions of the statistical model.

This result could have been anticipated directly from our basic equations (1.8) and (1.9). They can be interpreted as providing one resonance level for each scattering state that can get inside the interaction volume with unit phase space.[24]

An estimate[25] of the numerical coefficient in Eq. (5.6) yields $cm_0{}^{a+1}/[-(a+1)] \approx 10$. Thus one might be worried that channels are much more numerous than resonances even though they both rise with the same functional dependence on mass. But when one takes proper account of the states of motion in each channel, as in Eqs. (1.8) and (1.9), this discrepancy is removed.

VI. THERMODYNAMIC CONTENT OF MODEL

As we have seen, it is possible to set up a statistical model without reference to thermodynamics. However, now that a level density has been derived, we can write thermodynamic expressions in the usual way and study them. In the present section we review Hagedorn's remarkable results on this subject.

If thermodynamic equilibrium could be achieved in our box, the average energy would be

$$\overline{E} = \int_0^\infty dE\, E\rho(E)e^{-E/T} \bigg/ \int_0^\infty dE\, \rho(E)e^{-E/T} . \tag{6.1}$$

With the center of mass at rest, $E = m$ and the density of states is just $\rho(m) = cm^a e^{bm}$. This gives integrals of the form

$$\int_{m_0}^\infty dm\, cm^{a+1}e^{m(b-1/T)} , \tag{6.2}$$

which are defined only if $T < b^{-1}$. Thus, $b^{-1} \equiv T_0$ is a *maximum temperature*. To see what happens in more detail, consider the mathematical example $a = 0$, for which the integrals can be performed exactly, yielding

$$\overline{E} = m_0 + \frac{T_0 T}{T_0 - T} . \tag{6.3}$$

This result is plotted in Fig. 3. At $T = 0$, the energy in our box is just the ground-state rest mass m_0. As T increases to T_0, the average energy rises to infinity. Physically, what is happening is that as the energy in the box increases, it goes into the mass of new particles rather than into raising the kinetic energy of the existing particles.[26] In other words, the specific heat $C_V = dE/dT$ rises because of the many new particle modes the energy can go into.

Another property of the model at $T \approx T_0$ is the existence of large energy fluctuations.[10] Energy fluctuations are defined by

$$\frac{\Delta E}{E} = \left(\frac{\overline{E^2} - \overline{E}^2}{\overline{E}^2}\right)^{1/2}, \tag{6.4}$$

where

$$\overline{E^2} = \int_0^\infty dm\, m^2 \rho(m) e^{-m/T} \Big/ \int_0^\infty dm\, \rho(m) e^{-m/T} \tag{6.5}$$

is again easily calculated with our $\rho(m)$. One finds large $\Delta E/E \sim 1$ as T approaches T_0. The reason is easily seen. Normally $\rho(m)e^{-m/T}$ is a very strongly peaked function of mass, and $\overline{E}^2$ and $\overline{E^2}$ lie close together, near the peak. But in our case, as T approaches T_0, $\rho(m)e^{-m/T}$ is a slowly varying function of mass, allowing $\overline{E}^2$ and $\overline{E^2}$ to differ considerably. This is reminiscent of other situations in thermodynamics where a parameter is not sharply specified by T and exhibits large fluctuations; it often happens at a phase transition if some physical parameter is different in the two phases.

Hagedorn has made several important applications of the maximum-temperature concept. One application is to high-energy hadron reactions.[7-9] Each of the incoming hadrons is normally in its ground state (the lowest mass state for a particular baryon number, strangeness, and charge), and thus can be said to have zero internal temperature according to Eq. (6.3). During the collision, however, some of the incoming energy is converted into internal energy. In the original model of this type, Fermi's statistical model[5] for pp collisions, all of the incoming center-of-mass kinetic energy went into internal energy of an intermediate interaction volume. It was assumed that statistical equilibrium is reached in the volume; i.e., the particles in the volume could approximately be described as having a uniform nonzero temperature. The subsequent decay into final states then follows the statistical distribution in the box.

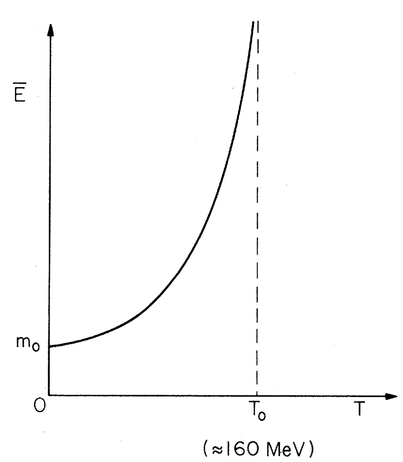

FIG. 3. Schematic plot of average energy $\overline{E}$ against temperature T for the case $\rho(m > m_0) = cm^a e^{bm}$, $a \geq -2$.

Fermi's model fails to fit high-energy data well. In particular, it fails to predict sufficiently strong forward-backward peaking in the c.m. system. Hagedorn and Ranft[9] removed this problem by assuming that the incoming particles retain a substantial fraction of their longitudinal momentum, converting the rest into internal energy. The internal energy is greatest at small impact parameters where "friction" is most intense, and less at large impact parameters. It is assumed that local thermodynamic equilibrium is achieved, the temperature being greatest at small impact parameters. Each portion of the incoming particles then decays into a distribution of final particles controlled by its local temperature.

At high energies the internal energies are quite high, and therefore correspond (Fig. 3) to temperatures near T_0. Thus secondary particles boil off with the weight factor $\exp[-(M^2 + p_\parallel^2 + p_\perp^2)^{1/2}T]$, T being close to but always below T_0. Hagedorn and collaborators[8,9] have made detailed fits on this basis; among the most impressive are the fits at large $p_\perp$ where the weight factor is approximately $\exp(-p_\perp/T) \approx \exp(-p_\perp/T_0)$, and the fits at large M (production of K pairs, $p\bar{p}$ pairs, etc.) where the factor is approximately $\exp(-M/T) \approx \exp(-M/T_0)$. From these fits, the value

$$b^{-1} \equiv T_0 \approx 160 \text{ MeV} \tag{6.6}$$

has been obtained. At present, these fits constitute the best numerical check on the model.

It is interesting to study how the thermodynamics depends on the power a in $\rho(m) \sim cm^a e^{bm}$. If a is not zero, evalutation of the integrals in $\overline{E}$ becomes more complicated, and the analog of Eq. (6.3) develops a branch point in the complex temperature plane at $T = T_0$, instead of the simple pole found in our example. However, the qualitative situation remains that pictured in Fig. 3, provided $a \geq -2$. If $a < -2$, the situation changes radically because the integrand of Eq. (6.2) favors low energies even at $T = T_0$. $\overline{E}$ rises with T as usual, but only to the limit

$$\lim_{T \to T_0;\, a < -2} \overline{E} = m_0 \frac{a+1}{a+2}, \tag{6.7}$$

which is $3m_0$ at $a = -\frac{5}{2}$, for example. This behavior is pictured schematically in Fig. 4. The exact value of the limit in Eq. (6.7) is not to be taken seriously, being sensitive to the low-mass spectrum which may differ from $cm^a e^{bm}$. But the qualitative behavior is that pictured in Fig. 4: There is a *maximum energy* as well as a maximum temperature above which thermodynamic equlibrium cannot be achieved.

Since $a < -\frac{5}{2}$ in our model, the picture envisioned by Fermi of high-energy collisions achieving a uniform temperature fails on theoretical as well as experimental grounds. However, Hagedorn's nonuniform temperature distribution is allowed provided $a \geq -\frac{7}{2}$. The reason is that if Eq. (6.1) is applied to each *local* region, the density of states includes d^3p_i for movement of the local region relative to the over-all c.m. system. The extra d^3p_i introduces a factor $m^{3/2}$ into $\rho_{\text{local}} \sim m^{a+3/2}e^{bm}$, and shifts the region where high-energy reactions can be described by thermodynamics to $a \geq -\frac{7}{2}$.[27]

Models with $a < -\frac{7}{2}$ are possible, though empirically somewhat unlikely in view of the success of Hagedorn's thermodynamic description of reactions. They are possible because they satisfy a "correspondence principle": There is a range of low temperatures and energies, including all temperatures at which thermodynamics has received laboratory checks, in which a *macroscopic* box containing hadrons with an $a < -\frac{7}{2}$ distribution will behave according to normal thermodynamics. The same cannot be said for distributions $\rho(m) \sim \exp(bm^p)$ with $p > 1$, yielded by some versions of the Veneziano model. These distributions do not allow the integrals in $\bar{E}$ to converge even at arbitrarily low temperatures, and thus do *not* satisfy the correspondence principle.

Another possible application of the high-mass hadron spectrum is to *macroscopic* states of very high density occurring in astrophysics, for example in the "big bang" and in the interior of neutron stars. Unfortunately, these applications face a serious difficulty: High-mass states become relevant precisely when the density is so high that hadrons are squeezing and overlapping each other in space. Under these conditions, our derivation of the hadron spectrum, which involved a box surrounded by high walls and isolated from outside influence, becomes questionable. Suppose, for example, that the neighboring hadrons squeezed the box to a smaller size. According to our self-consistency relations (2.32) and (2.34), the other parameters would have to adjust to the smaller volume V; the adjustment is not unique because of the presence of several parameters, but one possibility is a reduction in b (i.e., an increase in the limiting temperature T_0).

Nevertheless, Hagedorn[10] and Huang and Weinberg[19] have courageously assumed that the hadron spectrum is unchanged in a densely populated environment, and have studied the "big bang" theory. According to cosmology, the universe cools as it expands following the "big bang," from temperatures that exceeded 1 MeV during the first second. If the spectrum of particles did *not* rise exponentially, the temperature would rise higher and higher as we proceed back into the first second, eventually reaching the ionization point where matter dissociates into its quark constituents (if such a point exists). If there was once a phase when quarks dominated, quarks would still be quite numerous today because the subsequent cooling was too fast to allow all quarks to find each other and annihilate. Zeldovich[28] has estimated quarks would be about as common as gold, which is clearly contrary to observation.

If the spectrum rises like $\rho \sim e^{bm}$, a different scenario is obtained. The temperature never rises above 160 MeV, and quarks do not necessarily become numerous. This possibility has been worked out by Hagedorn[10] for the case $a = -\frac{5}{2}$, net baryon number zero, and by Huang and Weinberg[19] for more general cases including nonzero net baryon number.

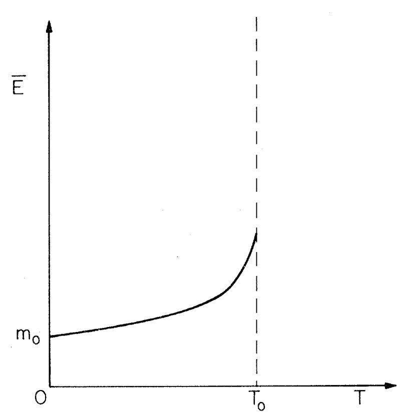

FIG. 4. Schematic plot of average energy $\bar{E}$ against temperature T for the case $\rho(m > m_0) = cm^a e^{bm}$, $a < -2$.

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APPENDIX: DENSITY OF STATES FOR $\rho_{\text{in}}(m) = \delta(m - m_0)$

If only a single variety of particle with mass m_0 is put into the box, in all possible combinations $n = 2, 3, \ldots \infty$, the density of states is

$$\rho(m) = \sum_{n=2}^{\infty} \left(\frac{V}{h^3}\right)^{n-1} \frac{1}{n!} \prod_{i=1}^{n} \int d^3p_i \, \delta\left(\sum_{i=1}^{n} E_i - m\right) \delta^3\left(\sum_{i=1}^{n} \vec{p}_i\right). \tag{1.6}$$

Consider first how one would evaluate (1.6) if the center-of-mass momentum were unrestricted and $m_0 = 0$ (if we multiplied by a spin factor 2^n, this would give the density of photon states in an enclosed volume). In the absence of $\delta^3(\vec{p}_{\text{c.m.}})$, all angular integrations can be done immediately and we obtain

$$\rho(m) = \sum_{n=2}^{\infty}\left(\frac{4\pi V}{h^3}\right)^n \frac{1}{n!}\int_0^m p_1^2 dp_1 \int_0^{m-p_1} p_2^2 dp_2 \cdots \int_0^{m-p_1-p_2-\cdots-p_{n-2}} p_{n-1}^2 dp_{n-1} \int_0^{m-p_1-p_2-\cdots-p_{n-1}} p_n^2 dp_n \delta\left(p_n + \sum_{i=1}^{n-1} p_i - m\right). \tag{A1}$$

The first integration, using the δ function, gives $(m - \sum_{i=1}^{n-1} p_i)^2$. All subsequent integrations have the form

$$\int_0^y dp_i p_i^2 (y - p_i)^r = 2y^{r+3}\frac{r!}{(r+3)!}. \tag{A2}$$

Working all the way down the chain, we obtain

$$\rho(m) = \sum_{n=2}^{\infty} \frac{1}{m}\left(\frac{8\pi m^3 V}{h^3}\right)^n \frac{1}{n!(3n-1)!}. \tag{A3}$$

The sum over all n can be approximated by an integral,

$$\rho(m) \approx \frac{1}{m}\int_0^{\infty} dn\, e^{f(n,m)}, \tag{A4}$$

where, with the aid of Stirling's formula,

$$f(n,m) \approx n \ln\left(\frac{8\pi m^3 V}{h^3}\right) - n\ln n + n - (3n-1)\ln(3n-1) + (3n-1). \tag{A5}$$

The maximum of f, where $\partial f/\partial n = 0$, occurs at

$$\bar{n} = \left(\frac{8\pi m^3 V}{27h^3}\right)^{1/4}, \tag{A6}$$

at which point

$$f_{\max} = 4\bar{n}. \tag{A7}$$

Thus, ρ grows as

$$\rho(m) \sim \exp\left[4\left(\frac{8\pi V}{27h^3}\right)^{1/4} m^{3/4}\right]. \tag{A8}$$

We digress briefly to note the consequence of Eq. (A6) for photons in thermal equilibrium in an enclosed volume: the temperature T goes like

$$kT = \langle \text{kinetic energy}\rangle_{\text{av}} = m/\bar{n} \propto m^{1/4}. \tag{A9}$$

Relabeling the total energy m by E, we obtain the familiar relation $E \propto T^4$.

Returning to the original problem, we find that $\rho \propto \exp(bm^{3/4})$ still holds for particles with mass $m_0 \neq 0$, because the average energy per particle grows like $m/\bar{n} \sim m^{1/4}$ and eventually becomes so large that the rest mass m_0 can be neglected. The behavior $\rho \propto \exp(bm^{3/4})$ is also unaffected by restricting the center-of-mass momentum to zero, because $\int d^3p_{\text{c.m.}}$ only contributes $\sim (m/n)^3$.

In the other cases considered in this paper, it is not possible to carry out the integrations exactly and approximations based on the optimum contribution from each particle must be made. It is reassuring to apply this procedure to Eq. (A1) and check that it yields essentially the same behavior as the exact integration of that equation. The integrand of Eq. (A1) is maximal when each p_i is of order m/n. Giving this value to each p_i, and not worrying much about numerical coefficients, we obtain the estimate

$$\rho(m) \sim \sum_{n=2}^{\infty}\left(\frac{4\pi V}{h^3}\right)^n \frac{1}{n!}\left(\frac{m}{n}\right)^{3n-1} = \frac{1}{m}\sum_{n=2}^{\infty}\left(\frac{4\pi m^3 V}{h^3}\right)^n \frac{1}{n!}\left(\frac{1}{n}\right)^{3n-1}, \tag{A10}$$

which again leads to $\exp(m^{3/4})$ growth of ρ.

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Simple Dual-Resonance Model for Inclusive Reactions*

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We write down a dual expression for the differential cross section in two-body reactions for inclusive production of a single particle with definite momentum. The formula is similar to the usual five-point amplitude but where the range of integration has changed. We show that it describes both limiting fragmentation and pionization. It furthermore shows approximate factorization as a function of $p_\perp^2$ and $p_\parallel^2$. The asymptotic behavior in $p_\perp^2$ is universal. We also generalize the formula for inclusive reactions where n particles are detected.

I. INTRODUCTION

Recently there has been considerable interest in the study of single-particle distributions in high-energy reactions. Several theoretical properties were predicted both from the parton model[1,2] and from the multiperipheral model.[3]

Some of the interesting features that have come out both from these models and from experiment are[4]: (i) The distribution in longitudinal momenta approaches a finite limit both in the lab frame (limiting fragmentation[1]) and in the c.m. frame (pionization). (ii) In the limiting-fragmentation region the differential cross section can be written approximately as

$$p_0 d\sigma/d^3p = f(p_\parallel^L)G(p_\perp^2), \tag{1}$$

where $G(p_\perp^2)$ is a universal function[4] (a decreasing exponential). (iii) In the limit of fast fragments, one should recover Regge behavior.

In this paper we want to show how the Veneziano amplitude can be used to derive an explicit formula for the differential cross sections which demonstrates all the properties stated above and